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Reliability-based design optimization using convex approximations and sequential optimization and reliability assessment method †

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Abstract

In this study, an effective method for reliability-based design optimization (RBDO) is proposed enhancing sequential optimization and reliability assessment (SORA) method by convex approximations. In SORA, reliability estimation and deterministic optimization are performed sequentially. The sensitivity and function value of probabilistic constraint at the most probable point (MPP) are obtained in the reliability analysis loop. In this study, the convex approximations for probabilistic constraint are constructed by utilizing the sensitivity and function value of the probabilistic constraint at the MPP. Hence, the proposed method requires much less function evaluations of probabilistic constraints in the deterministic optimization than the original SORA method. The efficiency and accuracy of the proposed method were verified through numerical examples.

Keywords: Convex approximations; Most probable point; Reliability-based design optimization; Sequential optimization and reliability assessment method

1. Introduction

In reliability-based design optimization (RBDO), uncertainty can be considered in the optimization process. Typical RBDO methods such as reliability index approach (RIA) and performance measure approach (PMA) have double-loop structures. Thus, single-loop single-vector (SLSV), sequential optimization and reliability assessment (SORA) [1], and other various methods have been proposed to convert the double-loop structure to single-loop or serial-loop in order to improve efficiency. Since reliability estimation and deterministic optimization are performed sequentially in SORA, the function evaluation of the probabilistic constraints is also required in the deterministic optimization [1]. In this study, convex approximations [2, 3] such as convex linearization (CONLIN), method of moving asymptotes (MMA), and globally convergent version of MMA (GCMMA) are constructed by utilizing the sensitivity and function value of the probabilistic constraint at the most probable point (MPP). Therefore, the proposed method requires much less function evaluations of probabilistic constraints in the deterministic optimization than the original SORA method. The accuracy and efficiency of the proposed method were verified through numerical examples.

2. Reliability-based design optimization

The formulation of RBDO problem is generally written as follows:

$$\begin{aligned} &\text{Min. } f(\mathbf{d},\mathbf{p}) \\ &\text{s.t } \Pr\left[g_j(\mathbf{x},\mathbf{p})>0\right]\le\Phi\left(-\beta_j^{\mathrm{t}}\right),\ j=1,\ldots,m,\ \mathbf{d}^L\le\mathbf{d}\le\mathbf{d}^U \end{aligned} \tag{1}$$

where $\mathbf{d}=\boldsymbol{\mu}(\mathbf{x})\in R^n$ is the design vector, and $\mathbf{d}^L$ and $\mathbf{d}^U$ are the lower and upper limits of the design vector, $\mathbf{x}\in R^{nr}$ is the random vector, $\mathbf{p}\in R^{np}$ are the fixed system parameters, and $\Phi(\cdot)$ is standard cumulative function of the normal distribution. Here, β_j^{t} is the corresponding target reliability index. SORA [1] was proposed to convert the double-loop structure into serial-loop. In SORA, reliability estimation and deterministic optimization are performed sequentially. If the estimated MPP is in the infeasible region, then boundary of violated constraint is shifted to the feasible direction, based on the MPP obtained in the previous iteration. Deterministic optimization is performed using original objective function and shifted constraints to find new design variables. These procedures are iterated until the convergence criteria are satisfied.

3. Proposed method

3.1 Convex approximations

Convex approximations are generated through sensitivity information and function value of the original function. For a given function, $f_j(\mathbf{d})$, the approximated function by the

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CONLIN [2] in the vicinity of $\mathbf{d}^{(k)}$ is formulated as follows:

$$\tilde{f}_j^{\mathrm{CONLIN}(k)}(\mathbf{d}) = f_j\left(\mathbf{d}^{(k)}\right) + \sum_{+,i} \left.\frac{\partial f_j}{\partial d_i}\right|_{\mathbf{d}^{(k)}} \left(d_i - d_i^{(k)}\right) + \sum_{-,i} \left.\frac{\partial f_j}{\partial d_i}\right|_{\mathbf{d}^{(k)}} \frac{d_i^{(k)}}{d_i}\left(d_i - d_i^{(k)}\right) \tag{2}$$

where the symbol $\sum_{+,i}$ and $\sum_{-,i}$ means the summation over the terms containing the positive and negative first-order derivatives, respectively. The Hessian matrix of $\tilde{f}_j^{\mathrm{CONLIN}(k)}(\mathbf{d})$ is positive semi-definite if the design variables are greater than zero.

For the MMA and GCMMA, the authors utilized the last version proposed by Svanberg [3]. In the MMA, two sets of new parameters, namely, the lower and upper asymptotes, $L_i^{(k)}$ and $U_i^{(k)}$, are introduced to adjust the convexity of the approximation [3]. The original function is approximated as

$$\begin{aligned}
&\tilde{f}_j^{\mathrm{MMA}(k)}(\mathbf{d}) = \sum_{i=1}^{n}\left(\frac{p_{ij}^{(k)}}{U_i^{(k)} - d_i} + \frac{q_{ij}^{(k)}}{d_i - L_i^{(k)}}\right) + r_j^{(k)} \\
&p_{ij}^{(k)} = \left(U_i^{(k)} - d_i^{(k)}\right)^2 \left(\left(\frac{\partial f_j\left(\mathbf{d}^{(k)}\right)}{\partial d_i}\right)^+ + \left(\frac{\partial f_j\left(\mathbf{d}^{(k)}\right)}{\partial d_i}\right)^- + \frac{\rho_j}{d_i^U - d_i^L}\right) \\
&q_{ij}^{(k)} = \left(d_i^{(k)} - L_i^{(k)}\right)^2 \left(\left(\frac{\partial f_j\left(\mathbf{d}^{(k)}\right)}{\partial d_i}\right)^+ + \left(\frac{\partial f_j\left(\mathbf{d}^{(k)}\right)}{\partial d_i}\right)^- + \frac{\rho_j}{d_i^U - d_i^L}\right) \\
&r_j^{(k)} = f_j\left(\mathbf{d}^{(k)}\right) - \sum_{i=1}^{n}\left(\frac{p_{ij}^{(k)}}{U_i^{(k)} - d_i^{(k)}} + \frac{q_{ij}^{(k)}}{d_i^{(k)} - L_i^{(k)}}\right)
\end{aligned} \tag{3}$$

where $0.9L_i^{(k)} + 0.1d_i^{(k)} \le d_i \le 0.9U_i^{(k)} + 0.1d_i^{(k)}$, $\rho_j = 10^{-5}$, and $\left(\frac{\partial f_j\left(\mathbf{d}^{(k)}\right)}{\partial d_i}\right)^+$ denotes the largest of the two numbers 0 and $\frac{\partial f_j\left(\mathbf{d}^{(k)}\right)}{\partial d_i}$, while $\left(\frac{\partial f_j\left(\mathbf{d}^{(k)}\right)}{\partial d_i}\right)^-$ denotes the largest of the two numbers 0 and $-\frac{\partial f_j\left(\mathbf{d}^{(k)}\right)}{\partial d_i}$. Since all ρ_j are positive values, $\tilde{f}_j^{\mathrm{MMA}(k)}(\mathbf{d})$ is strictly convex. Detailed method for updating $L_i^{(k)}$ and $U_i^{(k)}$ is explained in Ref. [3]. The CONLIN is obtained by letting $L_i^{(k)} = 0$ and $U_i^{(k)} \to +\infty$. Moreover, the MMA is reduced to a linear expansion if $L_i^{(k)} \to -\infty$ and $U_i^{(k)} \to +\infty$. In the GCMMA, an extension of the MMA, a non-monotonic parameter, $\rho_j^{(k,v)}$, is added to MMA in order to ensure the global convergence property of the approximation scheme [3]. Similar to MMA, $\tilde{f}_j^{\mathrm{GCMMA}(k,v)}(\mathbf{d})$ is also strictly convex.

3.2 Proposed RBDO method

As mentioned in Section 2, the function evaluation of shifted constraints is required in the deterministic optimization of SORA. The sensitivity and function value of the probabilistic constraint at the MPP are obtained in the previous iteration. The *j-th* shifted constraint g_j in the deterministic optimization at $\mathbf{d} = \mathbf{d}^{(k-1)}$ is expressed as follows:

$$\left.g_j\left(\mathbf{d} - \mathbf{s}_j^{(k-1)}\right)\right|_{\mathbf{d}=\mathbf{d}^{(k-1)}} = \left.g_j\left(\mathbf{d} - \left(\mathbf{d}^{(k-1)} - \mathbf{x}_{\mathbf{MPP}_j}^{(k-1)}\right)\right)\right|_{\mathbf{d}=\mathbf{d}^{(k-1)}} = g_j\left(\mathbf{x}_{\mathbf{MPP}_j}^{(k-1)}\right) \tag{4}$$

The constraint g_j can be expressed using the independent standard normal distributed random variable vector $\mathbf{u}$. The vector $\mathbf{u}$ is transformed from the original random variable vector. For example, if the random variables $\mathbf{x}$ are statistically independent and have normal distribution, then u_i is calculated as $u_i = (x_i - d_i)/\sigma_i$, where σ_i is the standard deviation of the corresponding random variable x_i. In this study, convex approximations in the vicinity of the MPP were constructed by utilizing the sensitivity and function value of the probabilistic constraint at the MPP.

The *j-th* shifted constraint near the corresponding MPP by the CONLIN method is approximated as follows:

$$\left.\tilde{g}_j^{\mathrm{CONLIN}}\left(\mathbf{d} - \mathbf{s}_j^{(k-1)}\right)\right|_{\mathbf{d}\approx\mathbf{d}^{(k-1)}} = \left.g_j\left(\mathbf{d} - \mathbf{s}_j^{(k-1)}\right)\right|_{\mathbf{d}=\mathbf{d}^{(k-1)}} + \sum_{+} \left.\frac{\partial g_j}{\partial d_i}\right|_{\mathbf{d}=\mathbf{d}^{(k-1)}} \left(d_i - d_i^{(k-1)}\right) + \sum_{-} \left.\frac{\partial g_j}{\partial d_i}\right|_{\mathbf{d}=\mathbf{d}^{(k-1)}} \frac{d_i^{(k-1)}}{d_i}\left(d_i - d_i^{(k-1)}\right) \tag{5}$$

In Eq. (5), the sensitivity is obtained by using the chain rule as follows:

$$\left.\frac{\partial g_j}{\partial d_i}\right|_{\mathbf{d}=\mathbf{d}^{(k-1)}} = \left.\frac{\partial g_j}{\partial u_{j_i}}\frac{\partial u_{j_i}}{\partial x_i}\frac{\partial x_i}{\partial d_i}\right|_{\mathbf{u_j}=\mathbf{u_j}^{(k-1)}} \text{ and } \mathbf{u_j}^{(k-1)} = \mathbf{u}_{\mathbf{MPP}_j}^{(k-1)} \tag{6}$$

where u_{j_i} is the independent standard normal distributed random variable vector corresponding to *j-th* constraint and *i-th* random design variable. In Eq. (6), $\partial u_{j_i}/\partial x_i$ and $\partial x_i/\partial d_i$ can be easily obtained by using the relation between $\mathbf{x}$ and $\mathbf{u}$. For example, if $\mathbf{x}$ are statistically independent and have normal distribution, then $\partial u_{j_i}/\partial x_i$ and $\partial x_i/\partial d_i$ is expressed as

$$x_i = d_i + \sigma_i u_i,\ \frac{\partial u_i}{\partial x_i} = \frac{1}{\sigma_i},\ \frac{\partial x_i}{\partial d_i} = 1 \tag{7}$$

Therefore, by using Eq. (6) and Eq. (7), Eq. (5) can be rewritten as

$$\begin{aligned}
\left.\tilde{g}_j^{\mathrm{CONLIN}}\left(\mathbf{d} - \mathbf{s}_j^{(k-1)}\right)\right|_{\mathbf{d}\approx\mathbf{d}^{(k-1)}} &= g_j\left(\mathbf{x}_{\mathbf{MPP}_j}^{(k-1)}\right) \\
&+ \sum_{+} \left.\frac{\partial g_j}{\partial u_{j_i}}\frac{\partial u_{j_i}}{\partial x_i}\frac{\partial x_i}{\partial d_i}\right|_{\mathbf{u_j}=\mathbf{u}_{\mathbf{MPP}_j}^{(k-1)}} \left(d_i - d_i^{(k-1)}\right) \\
&+ \sum_{-} \left.\frac{\partial g_j}{\partial u_{j_i}}\frac{\partial u_{j_i}}{\partial x_i}\frac{\partial x_i}{\partial d_i}\right|_{\mathbf{u_j}=\mathbf{u}_{\mathbf{MPP}_j}^{(k-1)}} \frac{d_i^{(k-1)}}{d_i}\left(d_i - d_i^{(k-1)}\right)
\end{aligned} \tag{8}$$

In Eq. (8), $g_j\left(\mathbf{x}_{\mathbf{MPP}_j}^{(k-1)}\right)$ and $\left.\frac{\partial g_j}{\partial u_{j_i}}\right|_{\mathbf{u_j}=\mathbf{u}_{\mathbf{MPP}_j}^{(k-1)}}$ is the function value and the sensitivity of the *j-th* probabilistic constraint at the corresponding MPP, respectively. The approximation by MMA is formulated as follows:

$$\tilde{g}_j^{\mathrm{MMA}}\left(\mathbf{d}-\mathbf{s}_j^{(k-1)}\right)\Big|_{\mathbf{d}\approx\mathbf{d}^{(k-1)}} = \sum_{i=1}^{n}\left(\frac{p_{ij}^{(k-1)}}{U_i^{(k-1)}-d_i}+\frac{q_{ij}^{(k-1)}}{d_i-L_i^{(k-1)}}\right)+r_j^{(k-1)}$$

$$p_{ij}^{(k-1)} = \left(U_i^{(k-1)}-d_i^{(k-1)}\right)^2\left(\sum_{+,i}\frac{\partial g_j}{\partial u_{j_i}}\frac{\partial u_{j_i}}{\partial x_i}\frac{\partial x_i}{\partial d_i}\Big|_{\mathbf{u_j}=\mathbf{u}_{\mathbf{MPP}_j}^{(k-1)}} + \sum_{-,i}\frac{\partial g_j}{\partial u_{j_i}}\frac{\partial u_{j_i}}{\partial x_i}\frac{\partial x_i}{\partial d_i}\Big|_{\mathbf{u_j}=\mathbf{u}_{\mathbf{MPP}_j}^{(k-1)}} + \frac{\rho_j}{d_i^U-d_i^L}\right)$$

$$q_{ij}^{(k-1)} = \left(d_i^{(k-1)}-L_i^{(k-1)}\right)^2\left(\sum_{+,i}\frac{\partial g_j}{\partial u_{j_i}}\frac{\partial u_{j_i}}{\partial x_i}\frac{\partial x_i}{\partial d_i}\Big|_{\mathbf{u_j}=\mathbf{u}_{\mathbf{MPP}_j}^{(k-1)}} + \sum_{-,i}\frac{\partial g_j}{\partial u_{j_i}}\frac{\partial u_{j_i}}{\partial x_i}\frac{\partial x_i}{\partial d_i}\Big|_{\mathbf{u_j}=\mathbf{u}_{\mathbf{MPP}_j}^{(k-1)}} + \frac{\rho_j}{d_i^U-d_i^L}\right)$$

$$r_j^{(k-1)} = g_j\left(\mathbf{x}_{\mathbf{MPP}_j}^{(k-1)}\right)-\sum_{i=1}^{n}\left(\frac{p_{ij}^{(k-1)}}{U_i^{(k-1)}-d_i^{(k-1)}}+\frac{q_{ij}^{(k-1)}}{d_i^{(k-1)}-L_i^{(k-1)}}\right) \quad (9)$$

Similar to MMA, the approximation by GCMMA can be also formulated by adding a non-monotonic parameter, $\rho_j^{(k,v)}$.

The flow chart of the proposed method is shown in Fig. 1. First, an initial design value is selected and inverse reliability analysis is performed in order to obtain the MPP for each probabilistic constraint. The sensitivity and function value of probabilistic constraint at MPP are obtained naturally in the inverse reliability analysis. Subsequently, convex approximations for each probabilistic constraint are constructed by utilizing the sensitivity and function value of the probabilistic constraint at the MPP. Thus, each probabilistic constraint is replaced by the convex approximated functions. Finally, deterministic optimization is performed by using the convex approximated functions. These procedures are iterated until the convergence criteria are satisfied.

For both CONLIN and MMA methods, no additional evaluation of the probabilistic constraint is required in constructing convex approximations. Additionally, only a few evaluations of the probabilistic constraint may be required for GCMMA by adding a non-monotonic parameter. Moreover, no additional evaluation of the probabilistic constraint is required in the deterministic optimization of SORA by using convex approximated functions.

4. Numerical examples

In order to verify the accuracy and the efficiency of the

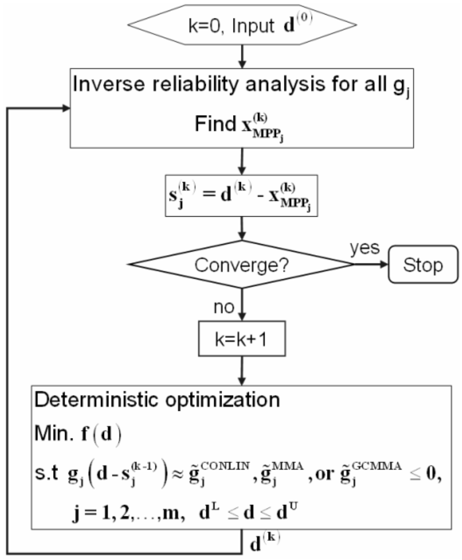

Fig. 1. Flow chart of the proposed method.

proposed RBDO methods (CONLIN, MMA, GCMMA), numerical examples were tested and compared to RIA, PMA, SLSV, and SORA.

4.1 Mathematical example 1

The first example [4] has ten random variables and eight probabilistic constraints. The description of this example is as follows:

$$\begin{aligned}
&\text{Min. } f(\mathbf{d}) = d_1^2+d_2^2+d_1d_2-14d_1-16d_2+(d_3-10)^2\\
&\qquad +4(d_4-5)^2+(d_5-3)^2+2(d_6-1)^2+5d_7^2\\
&\qquad +7(d_8-11)^2+2(d_9-10)^2+(d_{10}-7)^2+45\\
&\text{s.t } \Pr\left[g_i(\mathbf{x})>0\right]\le\Phi\left(-\beta_i^{\mathrm{t}}\right),\ i=1,\cdots,8\\
&\text{where,}\\
&g_1(\mathbf{x}) = \frac{4x_1+5x_2-3x_7+9x_8}{105}\text{-}1\ ,\\
&g_2(\mathbf{x}) = 10x_1-8x_2-17x_7+2x_8\ ,\\
&g_3(\mathbf{x}) = \frac{-8x_1+2x_2+5x_9-2x_{10}}{12}\text{-}1\ ,\\
&g_4(\mathbf{x}) = \frac{3(x_1-2)^2+4(x_2-3)^2+2x_3^2-7x_4}{120}\text{-}1,\\
&g_5(\mathbf{x}) = \frac{5x_1^2+8x_2+(x_3-6)^2-2x_4}{40}\text{-}1,\\
&g_6(\mathbf{x}) = \frac{0.5(x_1-8)^2+2(x_2-4)^2+3x_5^2-x_6}{30}-1,\\
&g_7(\mathbf{x}) = x_1^2+2(x_2-2)^2-2x_1x_2+14x_5-6x_6,\\
&g_8(\mathbf{x}) = -3x_1+6x_2+12(x_9-8)^2-7x_{10},\\
&\beta_1^{\mathrm{t}}=\beta_2^{\mathrm{t}}=\cdots=\beta_8^{\mathrm{t}}=3.0,\ d_i\ge 0 \text{ for } i=1,\cdots,10,\\
&x_i\sim N\left(d_i,0.02^2\right) \text{ for } i=1,\cdots,10,\\
&\mathbf{d}^{(0)}=\left[2.17,2.36,8.77,5.10,0.99,1.43,1.32,9.83,8.28,8.38\right]^{\mathrm{T}}
\end{aligned} \quad (10)$$

Table 1. Summary of the optimization results for Example 1.

Method	$f(\mathbf{d}^*)$	$\mathbf{d}^*$	No. of fun. call	
			f	g
RIA	27.75	(2.13, 2.34, 8.71, 5.10, 0.93, 1.47, 1.38, 9.80, 8.15, 8.48)	88	21736
PMA	27.75	(2.13, 2.34, 8.71, 5.10, 0.93, 1.46, 1.38, 9.81, 8.15, 8.47)	143	29032
SLSV	27.75	(2.14, 2.32, 8.71, 5.09, 0.92, 1.45, 1.39, 9.81, 8.15, 8.45)	101	9784
SORA	27.75	(2.14, 2.33, 8.71, 5.10, 0.93, 1.46, 1.39, 9.81, 8.15, 8.46)	151	1844
CONLIN	27.85	(2.13, 2.35, 8.71, 5.10, 0.93, 1.45, 1.38, 9.80, 8.13, 8.47)	154	612
MMA	27.81	(2.13, 2.34, 8.71, 5.10, 0.93, 1.45, 1.38, 9.80, 8.15, 8.48)	219	612
GCMMA	27.81	(2.13, 2.34, 8.71, 5.10, 0.93, 1.45, 1.38, 9.80, 8.15, 8.48)	219	628

Table 2. Evaluation of probabilistic constraints of Example 1.

Method	β_{MCS}^1	β_{MCS}^2	β_{MCS}^3	β_{MCS}^4	β_{MCS}^5	β_{MCS}^7
RIA	3.00	3.00	3.00	3.00	3.00	3.00
PMA	3.00	3.00	3.00	3.00	3.00	3.01
SLSV	3.00	3.00	3.01	3.04	3.00	∞
SORA	3.00	3.01	3.00	3.00	3.00	3.00
CONLIN	3.04	3.07	3.21	2.98	3.07	3.00
MMA	3.14	3.00	3.02	3.00	2.98	3.00
GCMMA	3.14	3.00	3.02	3.00	2.98	3.00

Table 3. Summary of the optimization results for the speed reducer.

Method	$f(\mathbf{d}^*)$	$\mathbf{d}^*$	No. of fun. call	
			f	g
RIA	3038.6	(3.58, 0.70, 17.00, 7.30, 7.75, 3.37, 5.30)	17	29096
PMA	3040.0	(3.58, 0.70, 17.00, 7.30, 7.76, 3.37, 5.30)	28	5852
SLSV	3048.5	(3.59, 0.70, 17.00, 7.30, 7.78, 3.37, 5.31)	17	1771
SORA	3040.0	(3.58, 0.70, 17.00, 7.30, 7.76, 3.37, 5.30)	36	1023
CONLIN	3040.6	(3.58, 0.70, 17.00, 7.30, 7.76, 3.37, 5.30)	36	627
MMA	3045.2	(3.59, 0.70, 17.00, 7.30, 7.76, 3.37, 5.30)	36	627
GCMMA	3045.4	(3.59, 0.70, 17.00, 7.30, 7.76, 3.37, 5.30)	36	649

All random variables are statistically independent and have normal distribution. The initial design point is selected as the result of deterministic optimization. The optimization results are summarized in Table 1 and 2. In Table 2, β_{MCS}^i stands for the reliability of the i-th probabilistic constraint at the optimum, which is evaluated by MCS with a ten-million sample size in order to confirm whether the target reliability of probabilistic constraints is satisfied. There are six active probabilistic constraints (g_1, g_2, g_3, g_4, g_5, g_7) at the optimum. From Table 1 and 2, all methods resulted in almost the same optimum and all the probabilistic constraints satisfied the target reliability. It is shown that RIA and PMA are not effective because of the double-loop structure. The efficiency of SLSV and SORA are more improved than the double-loop methods.

Though the number of objective function calls is increased, the number of probabilistic constraint function calls for the proposed methods is dramatically decreased. Results show that CONLIN is the most efficient. For GCMMA, addition of a non-monotonic parameter is required, thus the number of probabilistic constraint function calls is increased compared to CONLIN and MMA.

4.2 Speed reducer

A speed reducer is used to rotate the engine and propeller with efficient velocity in the light plane [1]. This problem has seven random variables and eleven probabilistic constraints. The description of the RBDO model of the speed reducer is as follows:

$$
\begin{aligned}
\text{Min } f(\mathbf{d}) = {} & 0.7854 d_1 d_2^2 \left(3.3333 d_3^2 + 14.9334 d_3 - 43.0934\right) \\
& + 1.508 d_1 \left(d_6^2 + d_7^2\right) + 7.477\left(d_6^3 + d_7^3\right) \\
& + 0.7854\left(d_4 d_6^2 + d_5 d_7^2\right)
\end{aligned}
$$

$$
\begin{aligned}
& \text{s.t } \Pr\left[g_i(\mathbf{x}) > 0\right] \le \Phi\left(-\beta_i^{\text{t}}\right),\ i = 1,\cdots,11 \\
& \text{where,} \\
& g_1(\mathbf{x}) = \frac{27}{x_1 x_2^2 x_3} - 1,\quad g_2(\mathbf{x}) = \frac{397.5}{x_1 x_2^2 x_3^2} - 1, \\
& g_3(\mathbf{x}) = \frac{1.93 x_4^3}{x_2 x_3 x_6^4} - 1,\quad g_4(\mathbf{x}) = \frac{1.93 x_5^3}{x_2 x_3 x_7^4} - 1, \\
& g_5(\mathbf{x}) = \frac{\sqrt{\left(\frac{745 x_4}{x_2 x_3}\right)^2 + 16.9\times 10^6}}{0.1 x_6^3} - 1100, \\
& g_6(\mathbf{x}) = \frac{\sqrt{\left(\frac{745 x_5}{x_2 x_3}\right)^2 + 157.5\times 10^6}}{0.1 x_7^3} - 850, \\
& g_7(\mathbf{x}) = x_1 x_3 - 40,\quad g_8(\mathbf{x}) = 5 - \frac{x_1}{x_2},\quad g_9(\mathbf{x}) = \frac{x_1}{x_2} - 12, \\
& g_{10}(\mathbf{x}) = \frac{1.5 x_6 + 1.9}{x_4} - 1,\quad g_{11}(\mathbf{x}) = \frac{1.1 x_7 + 1.9}{x_5} - 1, \\
& \beta_1^{\text{t}} = \beta_2^{\text{t}} = \cdots = \beta_{11}^{\text{t}} = 3.0, \\
& 2.6 \le d_1 \le 3.6,\ 0.7 \le d_2 \le 0.8,\ 17 \le d_3 \le 28, \\
& 7.3 \le d_4 \le 8.3,\ 7.3 \le d_5 \le 8.3,\ 2.9 \le d_6 \le 3.9, \\
& 5.0 \le d_7 \le 5.5, \\
& x_i \sim N\left(d_i, 0.005^2\right) \text{ for } i = 1,\cdots,7, \\
& \mathbf{d}^{(0)} = \left[3.5, 0.7, 17.0, 7.3, 7.72, 3.35, 5.29\right]^{\text{T}}
\end{aligned}
\tag{11}
$$

The objective function is minimizing the weight and probabilistic constraints are related to physical quantities. All random variables are statistically independent and have normal distribution. The initial design point is selected as the result of deterministic optimization. The optimization results are summarized in Table 3. The probabilistic constraints at the optimum are evaluated by MCS with a ten-million sample size and presented in Table 4. There are four active probabilistic constraints (g_5, g_6, g_8, g_{11}) at the optimum, and all the probabilistic constraints satisfy the target reliability at the optimum.

Table 4. Evaluation of probabilistic constraints of the speed reducer.

Method	β^{5}_{MCS}	β^{6}_{MCS}	β^{8}_{MCS}	β^{11}_{MCS}
RIA	3.02	3.00	3.00	3.02
PMA	3.15	3.08	3.07	4.22
SLSV	3.65	4.07	3.50	∞
SORA	3.15	3.08	3.07	4.21
CONLIN	3.14	3.08	3.14	4.23
MMA	3.21	3.17	3.57	3.45
GCMMA	3.38	3.15	3.58	3.56

Table 3 and 4 show that, double-loop methods such as RIA and PMA are not efficient, whereas, the efficiency of SLSV and SORA are improved more than the double-loop methods. Results therefore show that the proposed methods are the most efficient. The number of objective function calls for the proposed methods is equal to SORA, and the number of probabilistic constraint function calls for GCMMA is increased because of the addition of a non- monotonic parameter.

5. Conclusions

In this study, efficient RBDO method was proposed enhancing SORA method by convex approximations. In SORA, reliability estimation and deterministic optimization are performed sequentially. The sensitivity and function value of the probabilistic constraint at the MPP were evaluated in the reliability analysis loop. Convex approximations, such as CONLIN, MMA, and GCMMA, near the MPP were constructed by utilizing the sensitivity and function value of the probabilistic constraint at the MPP. Therefore, the proposed method requires much less function evaluations of probabilistic constraints in the deterministic optimization than the original SORA method. Two numerical examples were tested to verify the accuracy and efficiency of the proposed method. Results show the proposed method is highly efficient, especially in the estimation of probabilistic constraints with similar accuracy. Application of the proposed RBDO method to large structural problems remains to be studied as a future work.

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